Shape Interrogation II

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Introduction

- Polynomials are used in various branches of computational science.
- They can be found in mathematics, computer science, engineering and many other fields.
- There are two basic reasons for that:
  - Most functions can be approximated by polynomial functions, and
  - They are rather easy to use in a computer code.
- Thus, they serve as good substitutes for functions that are difficult to deal with.

- In this talk we will discuss some of their applications in Computer Aided Geometric Design and Geometric Modeling.

- In particular, we will discuss:

  - Polynomial systems and their solutions
  - Elements of elimination theory
  - Polynomial maps
  - Some Problems of this Area.

A Strange Example

- As an indication of the difference in moving from one dimension to the next, even for simple functions–like polynomials–let us consider the following:

- Example 1. Every polynomial function $y = p(x)$ with $p(x) > 0$, $\forall x \in$ R has at least one (real) critical point.

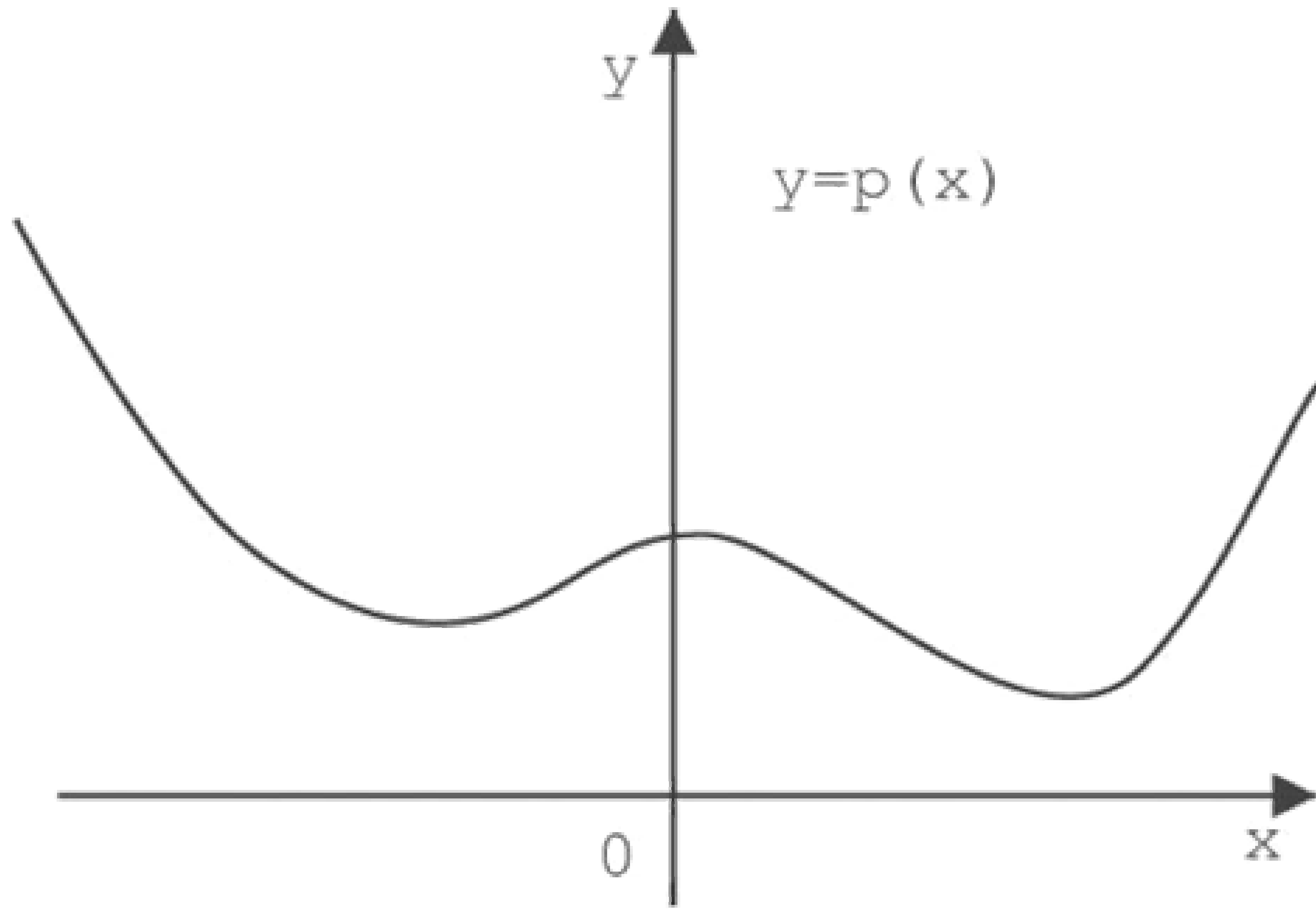
$$\left[\lim_{|x| \to \infty} p(x) = \infty.\right]$$

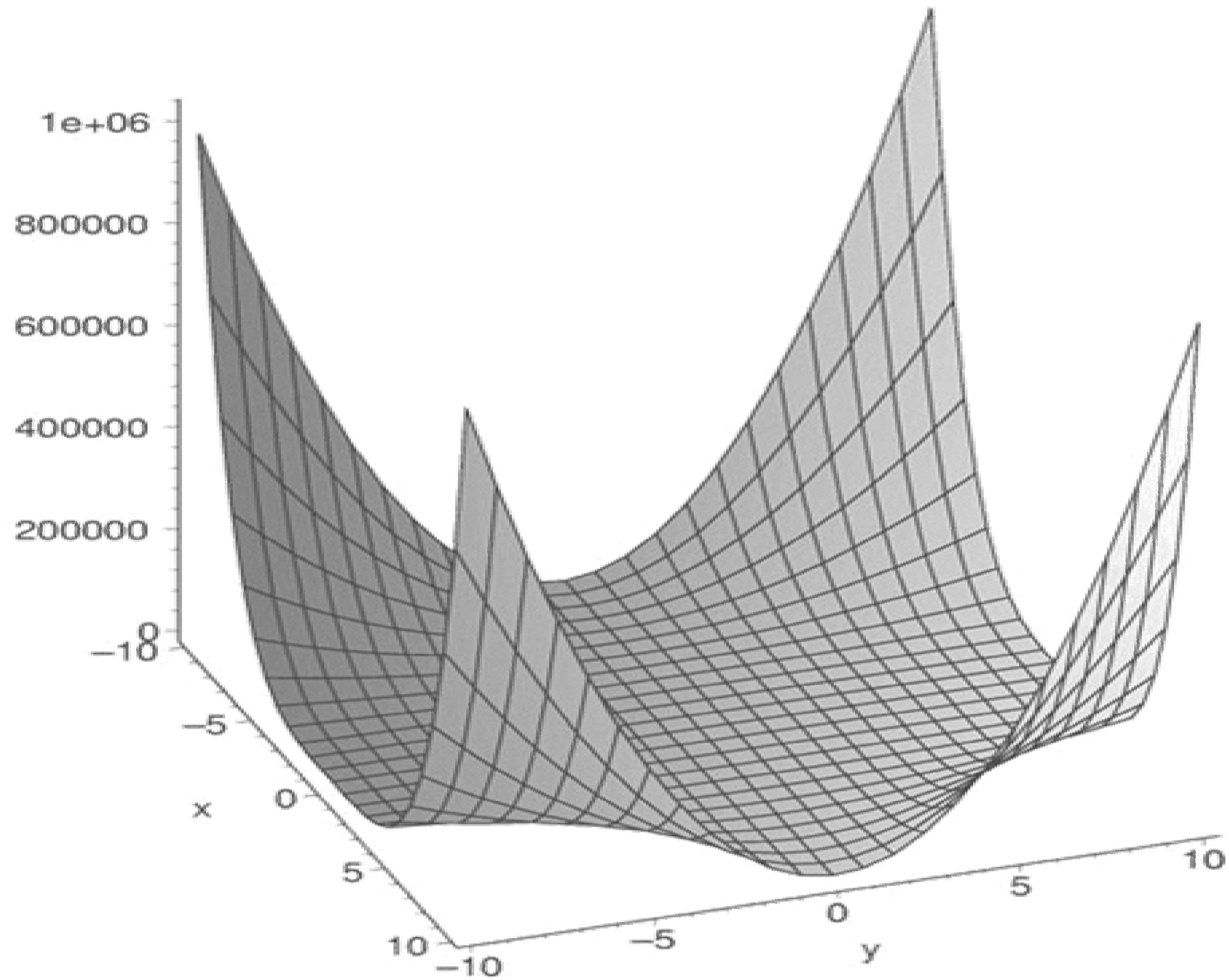
- Example 2. The polynomial

$$p(x,y) = (x^2 y - x - 1)^2 + x^2$$

- has the property that, for every $(x,y) \in$ R^2, $p(x,y) > 0$,
- but the function $p(x,y)$ does not have any (real) critical point.

$$\left[\lim_{|(x,y)| \to \infty} p(x,y) \text{ Does not exist.}\right]$$


y
y=p(x)
0
x



1e+06
800000
600000
400000
200000
0
−10
−5
0
5
10
x
−10
−5
0
5
10
y

Polynomial Systems

- Polynomials are popular in curve and surface representation.
- Thus, many critical problems in CAGD, such as surface interrogation, are reduced to finding the zero set of a system of polynomial equations

$$f(x) = 0$$

where $f = (f_1, \cdots, f_n)$ and each f_i is a polynomial of m independent variables $x = (x_1, \cdots, x_m)$.

Polynomial Systems

- Several root-finding methods for polynomial systems have been used in practice.
- These can be categorized as:
  – Algebraic and hybrid methods,
  – Homotopy methods, and
  – Subdivision methods.
- Among those types, the subdivision methods have been widely used in practice.
- The Interval Projected Polyhedral (IPP) algorithm is one example, and it has successfully been applied to various problems.

Motivation

- **Difficulties in handling roots with high multiplicity**
  - **Performance deterioration**
  - **Lack of robustness in numerical computation**
  - **Round-off errors during floating point arithmetic**

- **Limited research on root multiplicity of a system of equations**
  - **Heuristic approaches are needed for practical purposes.**

Objectives

- **Develop practical algorithms to isolate and compute roots and their multiplicities.**

- **Improve the Interval Projected Polyhedron (IPP) algorithms.**

Multiplicity of Roots

- **Univariate Case**
  - **A root a of *f(x)=0* has multiplicity *k* if**

$$f(a) = f^{'}(a) = \cdots = f^{(k-1)}(a) = 0, \; and \quad f^{(k)}(a) \neq 0$$

- **Bivariate Case**
  - **Define**

$$V_f = \{(x, y) \in \mathbf{C} \mid f(x, y) = 0\}$$

$$V_g = \{(x, y) \in \mathbf{C} \mid g(x, y) = 0\}$$

  - **Suppose that z_0 is the only common point of V_f and V_g lying above x_0. Consider $h(x)=Res_y(f,g)$, the resultant of *f,g* with respect to *y*. Then the multiplicity of $z_0=(x_0,y_0)$ as a root of the system is the multiplicity of x_0 as a zero of *h(x)*.**

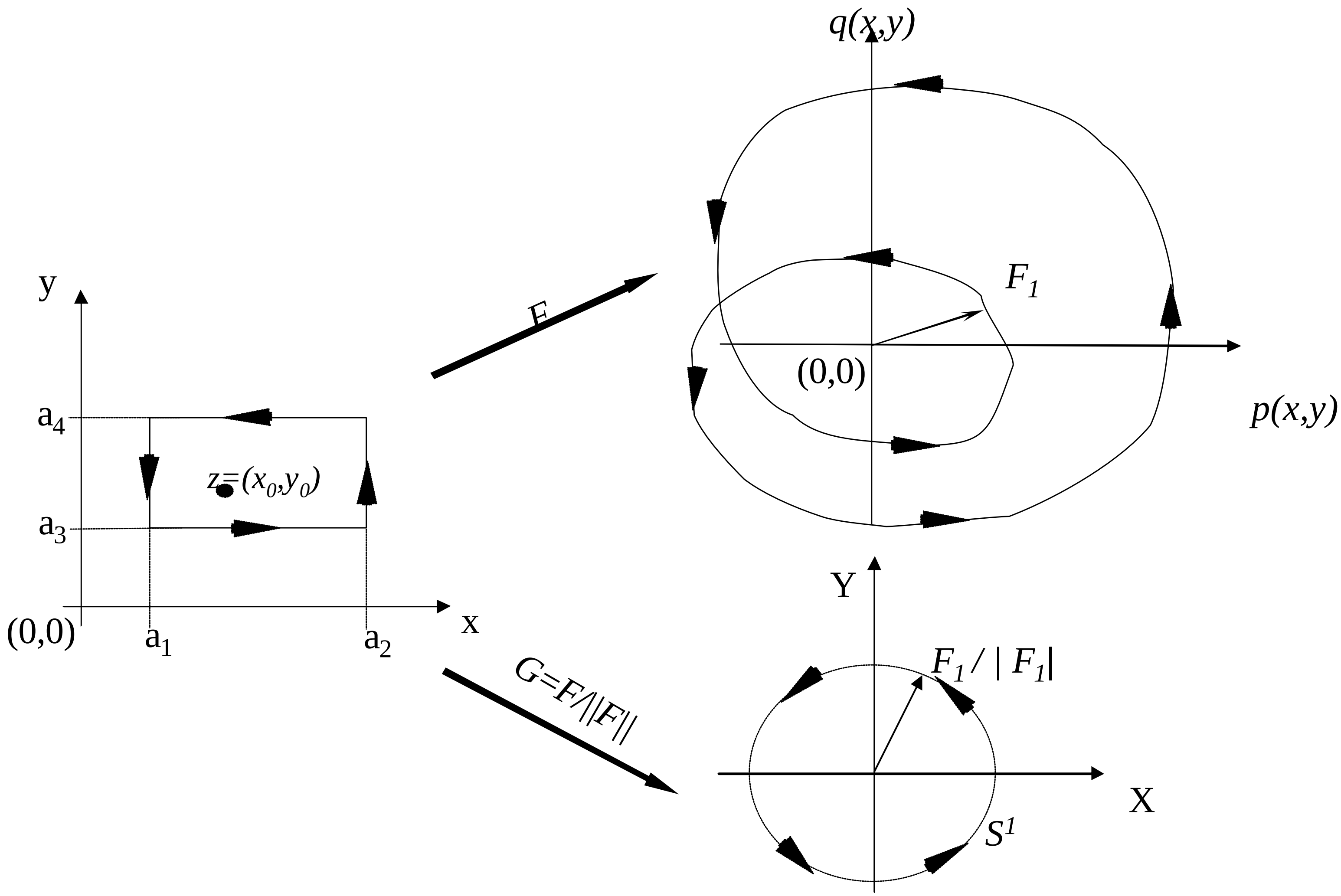
Degree of the Gauss Map

- **Let *p(x,y), q(x,y)* be polynomials with rational coefficients without common factors, of degrees n_1 *and* n_2, and let *F=(p, q).***

- **Let *A* be a rectangle in the plane defined by** $a_1 \le x \le a_2,\ a_3 \le y \le a_4,$ $a_1 < a_2,\ a_3 < a_4,\ a_i \in \mathbf{Q},\ i = 1,2,3,4$ **so that no zero of *F* lies its boundary** ∂A**, and** $p \cdot q$ **does not vanish at its vertices.**
  - Gauss map $G : \partial A \to S^1,\ G = F / \|F\|,$ where S^1 is the unit circle.
  - *G* is continuous ($\|F\| \neq 0 \ \ on \ \ \partial A$).
  - ∂A and S^1 carry the counterclockwise orientation.

- **Degree *d* of *G* : an integer indicating how many times** ∂A **is wrapped around S^1 by *G*.**

Illustration of the Gauss Map



The Cauchy Index

- **Preliminaries**
  - *R(x)* : a rational function *q(x)/p(x),* where *p, q* are polynomials.
  - [*a,b*] : a closed interval, *a* < *b*. *R* does not become infinite at the end points.

- **Definition of the *Cauchy index***

  By the *Cauchy index*, $I_a^b R$ of *R* over [*a,b*], we mean $I_a^b R = N_-^+ - N_+^-$ where $N_-^+ (N_+^-)$ denotes the number of points in (*a,b*) at which *R(x)* jumps from $-\infty \; to + \infty \; (+\infty \; to \; -\infty)$, respectively, as *x* is moving from *a* to *b*. Notice that $I_a^b R = -I_b^a R$ from the definition.

The Cauchy Index (continued)

- **Preliminaries**
  - *A* : a rectangle defined by [a_1, a_2] x [a_3, a_4] which encloses a zero.
  - *F = (p,q)* does not vanish on the boundary of *A*, ∂A.
  - $p \cdot q$ is not zero at each vertex of *A*.
  - Let

$$R_1 = \frac{q(a_1, y)}{p(a_1, y)},\ R_2 = \frac{q(a_2, y)}{p(a_2, y)},\ R_3 = \frac{q(x, a_3)}{p(x, a_3)},\ R_4 = \frac{q(x, a_4)}{p(x, a_4)}.$$

    Then, we set (for counterclockwise traversal of ∂A)

$$I_A F = I_{a_4}^{a_3} R_1 + I_{a_3}^{a_4} R_2 + I_{a_1}^{a_2} R_3 + I_{a_2}^{a_1} R_4.$$

- **Proposition***

•T. Sakkalis, "*The Euclidean Algorithm and the Degree of the Gauss Map*", SIAM J. Computing. Vol. 19, No. 3, 1990.

$I_A F$ is an even integer and the multiplicity $d = -\frac{1}{2} I_A F.$

Illustrative Example for Multiplicity Computation Using the Cauchy Index

- $p(x) = (x-1/2)^5 = 0$
- A root of $p(x)$, $[a] = [0.49, 0.51]$.
- $P(z)$; $(z = x+iy)$

$$p(z) = (x + iy - \frac{1}{2})^5 = f(x,y) + ig(x,y)$$

- Create

$$A = [0.49, 0.51] \times [-0.01, 0.01], \quad a_1 = 0.49,\ a_2 = 0.51,\ a_3 = -0.01,\ a_4 = 0.01$$

- Calculate the Cauchy index
  - Roots of $f(x, a_3) = 0$
  - Calculation of

$$I_{a_1}^{a_2} R_3 = -3$$

No.	Roots of $f(x,a_3) = 0$ in [0,1] (from the IPP)
1	[0.46922316412099, 0.46922316512099]
2	[0.49273457408967, 0.492734576204823]
3	[0.499999997363532, 0.500000001889623]
4	[0.507265424645288, 0.507265426808589]
5	[0.530776834861365, 0.530776835861365]

- Roots No. 2, 3, and 4 are selected since they lie within the interval $[a]$.

Illustrative Example (Continued)

- Similarly, $I_{a_3}^{a_4} R_2 = -2,\ I_{a_2}^{a_1} R_4 = 3,\ I_{a_4}^{a_3} R_1 = 2$

- Calculate $I_A F = I_{a_4}^{a_3} R_1 + I_{a_3}^{a_4} R_2 + I_{a_1}^{a_2} R_3 + I_{a_2}^{a_1} R_4 = -10$

- The multiplicity *m* of the root is $d = -\frac{1}{2} I_A F = 5$

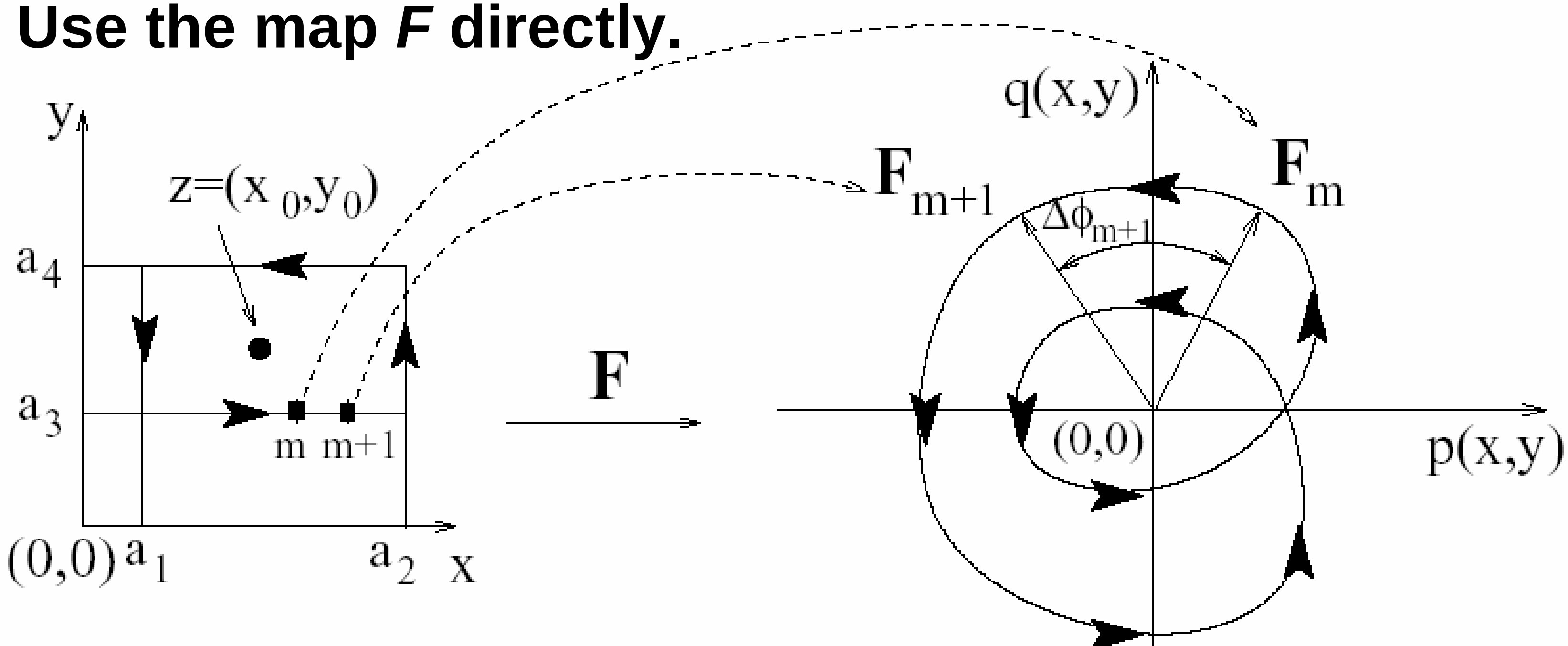
Note

– $I_a^b R = -I_b^a R.$

– *Counterclockwise orientation of* ∂A *is assumed.*

Direct Computation Method

- **Use the map *F* directly.**

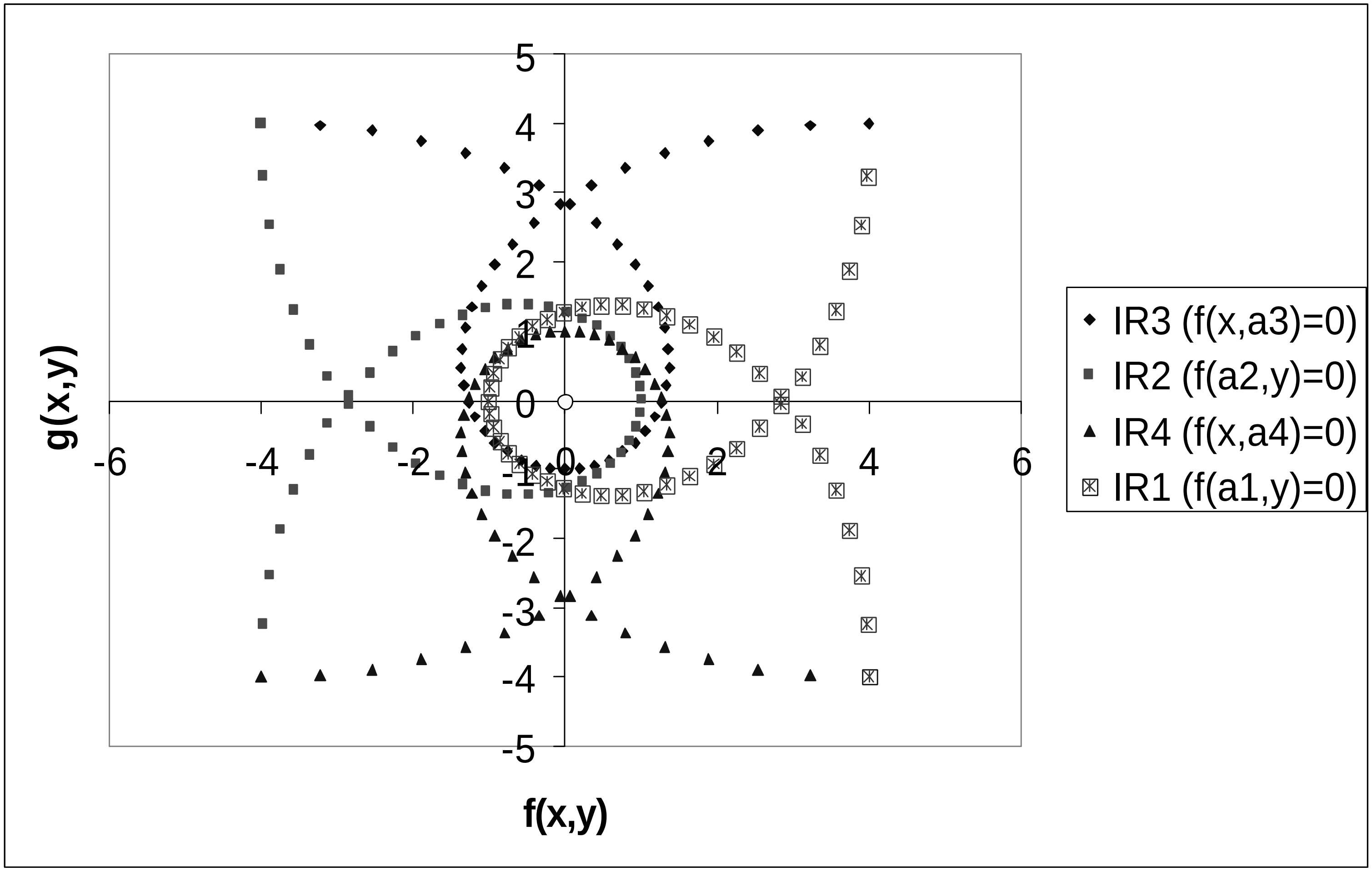


$$\phi_{total} = \sum_{i=0}^{n} \Delta\phi_{i+1}$$

$$d = \frac{\phi_{total}}{2\pi}$$

Direct Computation Method

$F:\mathbf{R}^2 \to \mathbf{R}^2,\ F(x,y) = (f(x,y), g(x,y)).\ G:\partial A \to S^1,\ G = F/\|F\|,$

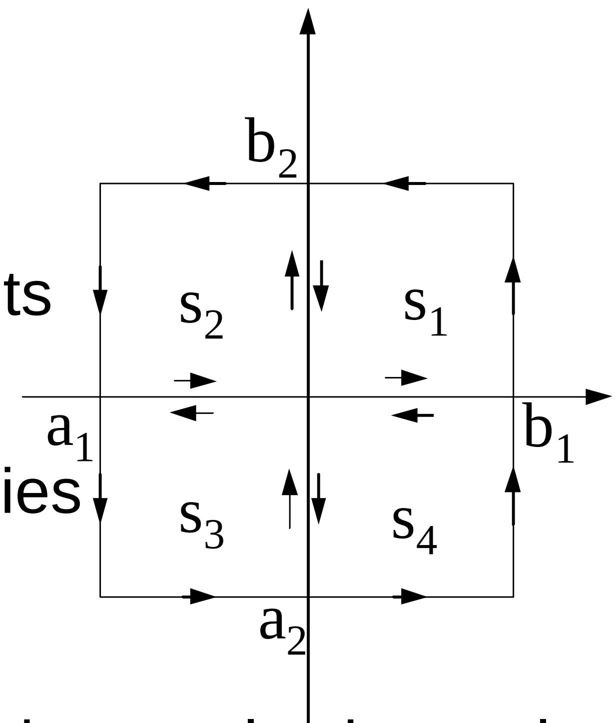


Bisection Algorithm for Solving Univariate Polynomial Equations

- **Univariate polynomial in complex variable *z*.**
  (Substitute x with a complex variable z = x+iy)

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z^1 + a_0 = 0$$

- **Input :**
  - initial domain : $S = [a_1, b_1] \times [a_2, b_2]$
  - a complex polynomial : *p(z)*
  - tolerance, number of sample points



- **Output**
  - real and complex roots, multiplicities
- **Algorithm**
  - Quadtree decomposition
  - Direct degree computation method : complex interval arithmetic.

Examples

- Wilkinson polynomial

$$p(t) = \prod_{i=1}^{20}\left(t - \frac{i}{20}\right)$$

No.	Multiplicity	Roots
1	1	[0.05,0.05]+i[-5.769e-10,5.769e-10]
2	1	[0.1,0.1]+i[-5.866e-10,5.866e-10]
3	1	[0.15,0.15]+i[-5.947e-10,5.947e-10]
4	1	[0.2,0.2]+i[-5.947e-10,5.947e-10]
5	1	[0.25,0.25]+i[-5.898e-10,5.898e-10]
6	1	[0.3,0.3]+i[-5.792e-10,5.792e-10]
7	1	[0.35,0.35]+i[-5.792e-10,5.792e-10]
8	1	[0.4,0.4]+i[-5.792e-10,5.792e-10]
9	1	[0.45,0.45]+i[-5.792e-10,5.792e-10]
10	1	[0.5,0.5]+i[-5.745e-10,5.745e-10]
11	1	[0.55,0.55]+i[-5.745e-10,5.745e-10]
12	1	[0.6,0.6]+i[-5.745e-10,5.745e-10]
13	1	[0.65,0.65]+i[-5.745e-10,5.745e-10]
14	1	[0.7,0.7]+i[-5.745e-10,5.745e-10]
15	1	[0.75,0.75]+i[-5.745e-10,5.745e-10]
16	1	[0.8,0.8]+i[-5.745e-10,5.745e-10]
17	1	[0.85,0.85]+i[-5.745e-10,5.745e-10]
18	1	[0.9,0.9]+i[-5.745e-10,5.745e-10]
19	1	[0.95,0.95]+i[-5.745e-10,5.745e-10]
20	1	[1,1]+i[-5.747e-10,5.747e-10]

- Complicated Polynomial (degree 22)

$$p(t) = (t^2+t+1)^2(t-1)^4 (t^3+t^2+t+1)^3(t-2)(t-4)^4$$

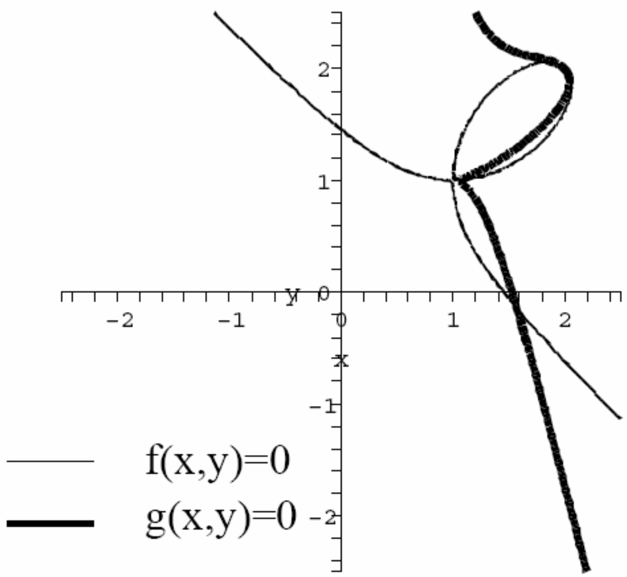
No.	Multiplicity	Roots
1	3	[-5.956e-10,5.956e-10]+i[1,1]
2	4	[1,1]+i[-5.956e-10,5.956e-10]
3	4	[4,4]+i[-5.939e-10,5.939e-10]
4	2	[-0.5,-0.5]+i[0.866,0.866]
5	3	[-1,-1]+i[-5.956e-10,5.956e-10]
6	2	[-0.5,-0.5]+i[-0.866,-0.866]
7	3	[-5.956e-10,5.956e-10]+i[-1,-1]
8	1	[2,2]+i[-5.94e-10,0]

Solving a Bivariate Polynomial System

- **Change of Coordinates**
  - **CR : *f* and *g* are regular in *y*.**
  - **CU : whenever two points (x_0,y_0) and (x_1,y_1) satisfy $f=g=0$, then $y_0=y_1$.**
- **Solving a Bivariate Polynomial System**
  - **Let f,g satisfy CR and CU and let $h(x)=Res_y(f,g)$. Then the roots of the system $f=g=0$ are in a one to one correspondence with the roots of $h(x)$. Moreover, $z_i=(x_i,y_i)$ is a real root if and only if x_i is a real root of $h(x)$.**
  - **Let $h(x)=Res_y(f,g)$ and $l(y)=Res_x(f,g)$ and $a_{ij}=[t_i,t_{i+1}]x[s_j,s_{j+1}]$ where in each subinterval $[t_i,t_{i+1}]$ or $[s_j,s_{j+1}]$ there exist precisely one root of $h(x)$ and $l(y)$, *respectively*. If a_{ij} encloses a real root of $f=g=0$, then the following must be true**

$$0 \in f([t_i,t_{i+1}],[s_j,s_{j+1}]) \times g([t_i,t_{i+1}],[s_j,s_{j+1}])$$

Solving a Bivariate Polynomial System : Example



$$
\begin{aligned}
f(x,y) &= x^3 - 3x^2 + 5x - 4 + y^3 \\
&\quad -3y^2 + 5y - 2xy = 0, \\
g(x,y) &= 2x^3 - 2x^2 + x - 4 - 4x^2y + 2xy \\
&\quad +9y + 3xy^2 - 8y^2 + y^3 = 0,
\end{aligned}
$$

$$
\begin{aligned}
h(x) &= 56x^9 - 704x^8 + 3880x^7 - 12304x^6 \\
&\quad +24744x^5 - 32736x^4 + 28504x^3 \\
&\quad -15760x^2 + 5024x - 704.
\end{aligned}
$$

$$
\begin{aligned}
l(y) &= -56y^9 + 608y^8 - 2824y^7 + 7312y^6 \\
&\quad -11496y^5 + 11136y^4 - 6328y^3 \\
&\quad +1744y^2 - 32y - 64.
\end{aligned}
$$

Root (x,y)	d
[0.999999978, 1.00000001]x[0.99999994, 1.00000001]	5
[1.57142855, 1.57142859]x[-0.142857209, -0.142857134]	1
[1.99999999, 2.00000003]x[1.99999996, 2.00000003]	3

Elimination Theory

I. Resultants
   - Sylvester Resultant
   - Macaulay Resultant
   - Sparse Resultant
   - D-Resultant

II. Groebner Bases

III. Symbolic System Solving

Elements of Resultant Theory

- Let:
$$a(t) = a_n t^n + \cdots + a_1 t + a_0$$
$$b(t) = b_m t^m + \cdots + b_1 t + b_0$$
- non zero polynomials, with complex coefficients.
- The resultant of a,b wrt t (or the t -resultant), $\mathrm{Res}_t(a,b) = R$ is

$$R = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & \cdots & a_1 & a_0 & & \\ & & \ddots & & & \ddots & \\ & & & a_n & \cdots & \cdots & a_0 \\ b_m & \cdots & \cdots & \cdots & b_0 & & \\ & \ddots & & & & \ddots & \\ & & b_m & \cdots & \cdots & \cdots & b_0 \end{vmatrix}$$

- Observe that $\mathrm{Res}_t(a,b) \in \mathrm{C}$.

Properties of the Resultant

- Let us see some well known properties of the resultant:
- **Property 1.** There exist polynomials $A(t), B(t) \in \mathrm{C}[t]$ of degrees respectively, $n' < m, m' < n$ so that

$$a(t)A(t) + b(t)B(t) = \mathrm{Res}_t(a,b)\ . \tag{1}$$

- **Property 2.** $\mathrm{Res}_t(a,b) = 0 \Longleftrightarrow a(t)$ and $b(t)$ have a common factor of positive degree.

- **Property 3.**
- Let , $a(x,y) = a_n y^n + a_{n-1}(x) y^{n-1} + \cdots + a_0(x)$

$$b(x,y) = \sum_{i=0}^{m} b_{m-i}(x) y^{m-i} \in k[y][x]$$

- with a_n or $b_m \in \boldsymbol{C}^*$, and consider $p(x) = Res_y(a,b)$. If x_0 is a root of $p(y)$, then there exists $y_0 \in \boldsymbol{C}$ with the property $a(x_0, y_0) = b(x_0, y_0) = 0$

Cramer's Rule

- Let $f(x,y), g(x,y) \in C[x,y]$ two nonconstant polynomials, and let be indeterminates.

- Consider

$$F : \mathrm{C}^2 \to \mathrm{C}^2, \quad F = (f, g)$$
$$A(x,u,v) = \mathrm{Res}_y(f - u, g - v),$$
$$B(y,u,v) = \mathrm{Res}_x(f - u, g - v)$$

- with $F(0,0) = (0,0)$.

Cramer's Rule

- **Theorem**[Cramer's Rule] F has a polynomial inverse if and only if:

$$A(x,u,v) = ax + A_0(u,v),$$

- and

$$B(y,u,v) = by + B_0(u,v), \quad with \;\; ab \neq 0$$

- Moreover, if

$$G(x,y) := \left(-\frac{A_0(x,y)}{a}, -\frac{B_0(x,y)}{b} \right),$$

- Then G is the inverse of F.
- In addition,

$$\deg F = \deg F^{-1}$$